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Permutation symmetry for theta functions[☆]

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ABSTRACT

This paper does for combinations of theta functions most of what Carlson (2004) [1] did for Jacobian elliptic functions. In each case the starting point is the symmetric elliptic integral R_F of the first kind. Its three arguments (formerly squared Jacobian elliptic functions but now squared combinations of theta functions) differ by constants. Symbols designating the constants can often be used to replace 12 equations by three with permutation symmetry (formerly in the letters c, d, n for the Jacobian case but now in the subscripts 2, 3, 4 for theta functions). Such equations include derivatives and differential equations, bisection and duplication relations, addition formulas (apparently new for theta functions), and an example of pseudoaddition formulas.

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1. Notation

Define three functions normalized to 1 at $z = 0$:

$$\phi_p(z) = \theta_p(z)/\theta_p(0), \quad p = 2, 3, 4, \quad (1.1)$$

and one more function that vanishes at $z = 0$:

$$\phi_1(z) = \theta_1(z)/\theta'_1(0), \quad \theta'_1(0) = \theta_2(0)\theta_3(0)\theta_4(0). \quad (1.2)$$

Dependence of all theta functions on the nome has been suppressed for brevity. Define three functions φ_{p1} and three functions φ_{1p} :

$$\varphi_{p1}(z) = \frac{\phi_p(z)}{\phi_1(z)} = \frac{1}{\varphi_{1p}(z)} = \frac{\theta_p(z)\theta'_1(0)}{\theta_p(0)\theta_1(z)}, \quad p = 2, 3, 4, \quad (1.3)$$

and six more functions φ_{pq} :

$$\varphi_{pq}(z) = \frac{\phi_p(z)}{\phi_q(z)} = \frac{1}{\varphi_{qp}(z)} = \frac{\theta_p(z)\theta_q(0)}{\theta_p(0)\theta_q(z)}, \quad p, q \in \{2, 3, 4\}, \quad p \neq q. \quad (1.4)$$

We now have 12 combinations of theta functions related by

$$\varphi_{pq} = \varphi_{p1}\varphi_{1q} = \varphi_{p1}/\varphi_{q1} = \varphi_{1q}/\varphi_{1p}, \quad (1.5)$$

and each of the 12 functions would acquire the name of a Jacobian elliptic function as a subscript if (2, 3, 4) were replaced by (c, d, n) and 1 by s. Those names reflect a connection shown in the next section.

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2. Connections and identities

2.1. Connections with Jacobi's elliptic functions and with an elliptic integral

Jacobi's elliptic functions $sn(u)$, $cn(u)$, and $dn(u)$, with modulus k omitted for brevity, can be expressed by [2, (2.1.1–3)] in terms of theta functions of variable z , where

$$u = \theta_3^2(0)z. \quad (2.1)$$

Rewriting theta functions in term of the phi functions of Section 1 then produces

$$sn(u) = \frac{\theta_3(0)\theta_1'(0)\phi_1(z)}{\theta_2(0)\theta_4(0)\phi_4(z)} = \theta_3^2(0)\varphi_{14}(z) = \frac{u}{z}\varphi_{14}(z), \quad (2.2)$$

$$cn(u) = \phi_2(z)/\phi_4(z) = \varphi_{24}(z), \quad (2.3)$$

$$dn(u) = \phi_3(z)/\phi_4(z) = \varphi_{34}(z). \quad (2.4)$$

Since $ns = 1/sn$, $ds = (dn)(ns)$, and $cs = (cn)(ns)$, it follows that

$$ns(u) = (z/u)\varphi_{41}(z), \quad ds(u) = (z/u)\varphi_{31}(z), \quad cs(u) = (z/u)\varphi_{21}(z). \quad (2.5)$$

Eqs. (2.5) are valid for any values of u and z that satisfy (2.1).

By [1, (1.8)] we have

$$u = R_F(cs^2(u), ds^2(u), ns^2(u)), \quad (2.6)$$

where R_F is the symmetric incomplete elliptic integral of the first kind. It is not only symmetric in its three variables but also homogeneous of degree $-1/2$. Therefore, when the variables are expressed by (2.5), a factor $(z/u)^2$ can be removed from each variable of R_F if a factor u/z is put in front of R_F . The u in the numerator cancels the u on the left side of (2.6), leaving

$$z = R_F(\varphi_{21}^2(z), \varphi_{31}^2(z), \varphi_{41}^2(z)). \quad (2.7)$$

Since (2.7) is the same as (2.6) except for a change of symbols, we change [1, (1.9)]:

$$\Delta(p, q) = ps^2(u) - qs^2(u) = -\Delta(q, p), \quad p, q \in \{c, d, n\}, \quad (2.8)$$

in the same way by defining

$$\Delta_{pq} = \varphi_{p1}^2(z) - \varphi_{q1}^2(z) = -\Delta_{qp}, \quad p, q \in \{2, 3, 4\}. \quad (2.9)$$

From (2.9), (2.5), and (2.8) we find

$$\Delta_{43} = \varphi_{41}^2 - \varphi_{31}^2 = (u/z)^2 \Delta(n, d), \quad (2.10)$$

$$\Delta_{42} = \varphi_{41}^2 - \varphi_{21}^2 = (u/z)^2 \Delta(n, c), \quad (2.11)$$

$$\Delta_{32} = \varphi_{31}^2 - \varphi_{21}^2 = (u/z)^2 \Delta(d, c). \quad (2.12)$$

By (2.1) and [1, (1.10)] these become

$$\Delta_{43} = \theta_3^4(0)k^2, \quad \Delta_{42} = \theta_3^4(0), \quad \Delta_{32} = \theta_3^4(0)(1 - k^2) \quad (2.13)$$

and finally, by [2, (2.1.7)],

$$\Delta_{43} = \theta_4^4(0), \quad \Delta_{42} = \theta_3^4(0), \quad \Delta_{32} = \theta_4^4(0). \quad (2.14)$$

Furthermore, the three equations in (2.14) are equivalent to [2, (1.4.50–52)] in the same order.

Example 2.1. By (2.9) and (1.3), $\Delta_{43} = \theta_4^4(0)$ becomes $\phi_4^2(z) - \phi_3^2(z) = \theta_4^4(0)\phi_1^2(z)$, and multiplication by $\theta_3^2(0)\theta_4^2(0)$ produces $\theta_3^2(0)\theta_4^2(z) - \theta_4^2(0)\theta_3^2(z) = \theta_2^2(0)\theta_1^2(z)$, which is [2, (1.4.50)]. See also [3, p. 466], [5, p. 323, Eqs. 8–10 or p. 336, Ex. 2], and [4, p. 53].

2.2. Three identities

Eqs. (2.14) will be applied to three algebraic identities to give new proofs of three well-known relations between theta functions.

If the identity $\phi_4^2 - \phi_2^2 = (\phi_4^2 - \phi_3^2) + (\phi_3^2 - \phi_2^2)$ is divided by ϕ_1^2 , it becomes $\Delta_{42} = \Delta_{43} + \Delta_{32}$, and by (2.14) this is

$$\theta_3^4(0) = \theta_2^4(0) + \theta_4^4(0), \quad (2.15)$$

which is [2, (1.4.53)].

More generally, if the identity

$$(\phi_4^2 - \phi_2^2)\phi_3^2 = (\phi_4^2 - \phi_3^2)\phi_2^2 + (\phi_3^2 - \phi_2^2)\phi_4^2 \quad (2.16)$$

is divided by ϕ_1^2 , it becomes

$$\Delta_{42}\phi_3^2 = \Delta_{43}\phi_2^2 + \Delta_{32}\phi_4^2. \quad (2.17)$$

By (2.14) and (1.1) this is

$$\theta_3^4(0)\theta_3^2(z)/\theta_3^2(0) = \theta_2^4(0)\theta_2^2(z)/\theta_2^2(0) + \theta_4^4(0)\theta_4^2(z)/\theta_4^2(0) \quad (2.18)$$

and therefore

$$\theta_3^2(0)\theta_3^2(z) = \theta_2^2(0)\theta_2^2(z) + \theta_4^2(0)\theta_4^2(z), \quad (2.19)$$

in agreement with [5, p. 323, (11)] and [2, (1.4.49)], of which (2.15) is the special case $z = 0$.

A more complicated procedure of this type proves an equation in [3, p. 469, Ex. 4], [5, p. 336, Ex. 3], and [2, p. 21, Exs. 6 and 10] that will be cited in Example 4.1. In the identity

$$(A - B)(A - C)(B - C) = (A - B)C^2 - (A - C)B^2 + (B - C)A^2, \quad (2.20)$$

let $A = \varphi_{41}^2$, $B = \varphi_{31}^2$, $C = \varphi_{21}^2$, where the argument z is suppressed for brevity. By (2.9) and (2.14) we find $A - B = \Delta_{43} = \theta_2^4(0)$, $A - C = \Delta_{42} = \theta_3^4(0)$, $B - C = \Delta_{32} = \theta_4^4(0)$, and thus (2.20) becomes $(\theta_2(0)\theta_3(0)\theta_4(0))^4 = (\theta_2(0)\varphi_{21})^4 - (\theta_3(0)\varphi_{31})^4 + (\theta_4(0)\varphi_{41})^4$. Multiplication by ϕ_1^4 produces $\theta_1^4 = \theta_2^4(0)\phi_2^4 - \theta_3^4(0)\phi_3^4 + \theta_4^4(0)\phi_4^4$, and by (1.1) this is

$$\theta_1^4(z) = \theta_2^4(z) - \theta_3^4(z) + \theta_4^4(z) = \sum_{p=2}^4 (-1)^p \theta_p^4(z). \quad (2.21)$$

The case $z = 0$ reduces to (2.15) because $\theta_1(0) = 0$.

3. Derivatives and differential equations

3.1. Derivatives

Proposition 3.1. *If $\{p, q, r\} = \{2, 3, 4\}$, a prime indicates differentiation with respect to z , and the variable z is suppressed for brevity, then*

$$\varphi'_{p1} = -\varphi_{q1}\varphi_{r1}, \quad \varphi'_{1p} = \varphi_{qp}\varphi_{rp}, \quad \varphi'_{pq} = \Delta_{pq}\varphi_{rq}\varphi_{1q}. \quad (3.1)$$

Proof. Eqs. (2.6) and (2.7) differ only by a change of symbols, and in each case all three variables of R_F have the same derivative because their differences are constants by (2.9) and (2.14). Thus [1, (2.1)] and its proof from (2.6) are converted by the same change of symbols into (3.1) and its proof from (2.7). \square

The first of Eqs. (3.1) includes [2, (1.9.11, 14, 16)], the second includes [2, (1.9.3, 8, 10)], and the third includes [2, (1.9.6, 7, 9, 12, 13, 15)].

Example 3.1A. If $p = 4$ the second equation in (3.1) becomes $(\phi_1/\phi_4)' = \phi_2\phi_3/\phi_4^2$ by (1.3). Substitution from (1.1) and (1.2) produces

$$(\theta_4(0)/\theta_1'(0))(\theta_1(z)/\theta_4(z))' = (\theta_2(z)/\theta_2(0))(\theta_3(z)/\theta_3(0))(\theta_4(0)/\theta_4(z))^2,$$

which becomes [3, p. 478, line 4] and [2, (1.9.3)] after cancellations.

Example 3.1B. If $p = 2$ and $q = 3$, the third equation in (3.1) becomes

$$(\phi_2/\phi_3)' = \Delta_{23}\phi_4\phi_1/\phi_3^2 \quad \text{by (1.4).}$$

Substitution from (1.1), (1.2), and (2.14) produces $(\theta_3(0)/\theta_2(0))(\theta_2(z)/\theta_3(z))' = -\theta_4^4(0)(\theta_4(z)/\theta_4(0))(\theta_1(z)/\theta_1'(0))(\theta_3(0)/\theta_3(z))^2$, which becomes [2, (1.9.9)] after cancellations.

3.2. Differential equations

Proposition 3.2. If $\{p, q, r\} = \{2, 3, 4\}$, a prime indicates differentiation with respect to z , and the variable z is suppressed for brevity, then

$$(\varphi'_{p1})^2 = (\Delta_{qp} + \varphi_{p1}^2)(\Delta_{rp} + \varphi_{p1}^2), \quad (3.2)$$

$$(\varphi'_{1p})^2 = (1 + \Delta_{qp}\varphi_{1p}^2)(1 + \Delta_{rp}\varphi_{1p}^2), \quad (3.3)$$

$$(\varphi'_{pq})^2 = (1 - \varphi_{pq}^2)(\Delta_{rp} + \Delta_{qr}\varphi_{pq}^2). \quad (3.4)$$

Proof. By (3.1), $(\varphi'_{p1})^2 = \varphi_{q1}^2\varphi_{r1}^2$, and (2.9) shows that each factor on the right-hand side is equal to the corresponding factor in (3.2). Since $\varphi_{1p} = 1/\varphi_{p1}$, we have $\varphi'_{1p} = -\varphi'_{p1}/\varphi_{p1}^2$ and $(\varphi'_{1p})^2 = (\varphi'_{p1})^2/\varphi_{p1}^4$, showing that (3.3) follows from (3.2). By (3.1), $(\varphi'_{pq})^2 = (\Delta_{pq}\varphi_{1q}^2)(\Delta_{pq}\varphi_{r1}^2)$, and by (2.9) the first product on the right-hand side is $(\varphi_{p1}^2 - \varphi_{q1}^2)\varphi_{1q}^2 = \varphi_{pq}^2 - 1$. For the second product we use the identity $\Delta_{pq}\varphi_{r1}^2 + \Delta_{rp}\varphi_{q1}^2 + \Delta_{qr}\varphi_{p1}^2 = 0$, a version of (2.17) that is easily verified by applying (2.9) to each Δ . Multiplying this identity by φ_{1q}^2 produces $\Delta_{pq}\varphi_{r1}^2 = -(\Delta_{rp} + \Delta_{qr}\varphi_{pq}^2)$, and multiplication of this second product by the first product proves (3.4). \square

Example 3.2. If $p = 4$ and we define $\xi = \theta_1(z)/\theta_4(z)$, then $\varphi_{14} = \xi/(\theta_2(0)\theta_3(0))$ by (1.3) and (1.2). Thus the left side of (3.3) becomes $(\xi')^2/(\theta_2^2(0)\theta_3^2(0))$. By (2.14) one factor on the right-hand side of (3.3) is

$$1 - \frac{\theta_2^4(0)\xi^2}{\theta_2^2(0)\theta_3^2(0)} = \frac{\theta_3^2(0) - \theta_2^2(0)\xi^2}{\theta_3^2(0)},$$

and the other factor is the same with subscripts 2 and 3 interchanged. The denominators of the two sides of (3.3) now cancel, leaving [3, p. 478, line 6], and [2, (1.9.19)]. Since (3.3) was derived from (3.2), this example indirectly supports (3.2). Also, (3.2) has been used to check the second equation in [2, p. 22, Ex. 12].

4. Bisection and duplication

4.1. Bisection

Proposition 4.1. If $\{p, q, r\} = \{2, 3, 4\}$ and the variable z is suppressed for brevity except when divided by 2, then

$$\varphi_{p1}^2(z/2) = (\varphi_{p1} + \varphi_{q1})(\varphi_{p1} + \varphi_{r1}) = \varphi_{p1}^2(1 + \varphi_{qp})(1 + \varphi_{rp}), \quad (4.1)$$

$$\varphi_{1p}^2(z/2) = \frac{1}{(\varphi_{p1} + \varphi_{q1})(\varphi_{p1} + \varphi_{r1})} = \frac{\varphi_{1p}^2}{(1 + \varphi_{qp})(1 + \varphi_{rp})}, \quad (4.2)$$

$$\varphi_{pq}^2(z/2) = \frac{\varphi_{p1} + \varphi_{r1}}{\varphi_{q1} + \varphi_{r1}} = \frac{\varphi_{pr} + 1}{\varphi_{qr} + 1}. \quad (4.3)$$

Proof. Since (2.1)–(2.5) remain valid if both u and z are multiplied by the same constant, we have $ns(u/2) = (z/u)\varphi_{41}(z/2)$ and similar modifications of the two other equations in (2.5). Substitution of these modifications of (2.5) changes [1, (3.1)]:

$$ps^2(u/2) = (ps + qs)(ps + rs) = ps^2(1 + qp)(1 + rp)$$

to (4.1) after cancellation of factors (z/u) . Then (4.2) and (4.3) follow from $\varphi_{1p}^2 = 1/\varphi_{p1}^2$ and $\varphi_{pq}^2 = \varphi_{p1}^2/\varphi_{q1}^2$, respectively. \square

4.2. Duplication

Proposition 4.2. If $\{p, q, r\} = \{2, 3, 4\}$ and the variable z is suppressed for brevity except when multiplied by 2, then

$$\varphi_{p1}(2z) = \frac{\varphi_{p1}^4 - \Delta_{pq}\Delta_{pr}}{2\varphi_{p1}\varphi_{q1}\varphi_{r1}} = \frac{1 - \Delta_{pq}\Delta_{pr}\varphi_{1p}^4}{2\varphi_{1p}\varphi_{qp}\varphi_{rp}}, \quad (4.4)$$

$$\varphi_{1p}(2z) = \frac{2\varphi_{p1}\varphi_{q1}\varphi_{r1}}{\varphi_{p1}^4 - \Delta_{pq}\Delta_{pr}} = \frac{2\varphi_{1p}\varphi_{qp}\varphi_{rp}}{1 - \Delta_{pq}\Delta_{pr}\varphi_{1p}^4}, \quad (4.5)$$

$$\varphi_{pq}(2z) = \frac{\varphi_{p1}^4 - \Delta_{pq}\Delta_{pr}}{\varphi_{q1}^4 - \Delta_{qp}\Delta_{qr}} = \frac{\varphi_{pq}^4 - \Delta_{pq}\Delta_{pr}\varphi_{1q}^4}{1 - \Delta_{qp}\Delta_{qr}\varphi_{1q}^4}. \quad (4.6)$$

Proof. Each of the three equations in (2.5) has the form $ps(u) = (z/u)\varphi_{p1}(z)$, where ps stands for cs , ds , or ns when p on the right-hand side stands for 2, 3, or 4, respectively. This unified form of (2.5) remains valid if both u and z are multiplied by the same constant (for example, by 2). Also, it follows from (2.8) and (2.9) that $\Delta(p, q) = (z/u)^2\Delta_{pq}$. Thus [1, (4.3)]:

$$ps(2u) = \frac{ps^4 - \Delta(p, q)\Delta(p, r)}{2(ps)(qs)(rs)}, \quad \{p, q, r\} = \{c, d, n\},$$

becomes, after cancellation of factors (z/u) ,

$$\varphi_{p1}(2z) = \frac{\varphi_{p1}^4 - \Delta_{pq}\Delta_{pr}}{2\varphi_{p1}\varphi_{q1}\varphi_{r1}}, \quad \{p, q, r\} = \{2, 3, 4\}.$$

This equation is the first equation in (4.4), and the first equation in (4.5) or (4.6) follows from it by $\varphi_{1p} = 1/\varphi_{p1}$ or $\varphi_{pq} = \varphi_{p1}/\varphi_{q1}$, respectively. \square

From (2.9) and (2.14) it follows that $\Delta_{pq}\Delta_{pr} = \pm\theta_r^4(0)\theta_q^4(0)$ with $+$ if $p = 4$ or 2 and $-$ if $p = 3$. Thus

$$\Delta_{pq}\Delta_{pr} = (-1)^p\theta_r^4(0)\theta_q^4(0), \quad \{p, q, r\} = \{2, 3, 4\}. \quad (4.7)$$

Proposition 4.3. In [3, p. 488, Exs. 4, 5] and [2, p. 21, Ex. 10] the equation for $\theta_1(2z)$ is already symmetric in subscripts 2, 3, 4, and the other three equations can be unified in the form

$$\theta_p(2z)\theta_p^3(0) = \theta_p^4(z) - (-1)^p\theta_1^4(z), \quad p = 2, 3, 4. \quad (4.8)$$

Proof. By (4.4) and (4.7) we have

$$\frac{\phi_p(2z)}{\phi_1(2z)} = \frac{(\phi_p/\phi_1)^4 - (-1)^p\theta_r^4(0)\theta_q^4(0)}{2\phi_p\phi_q\phi_r/\phi_1^3}, \quad (4.9)$$

where the variable z is suppressed except when multiplied by 2. On the right-hand side, multiplication of numerator and denominator by ϕ_1^4 produces

$$\frac{\phi_p(2z)}{\phi_1(2z)} = \frac{\phi_p^4 - (-1)^p(\theta_r(0)\theta_q(0)\phi_1)^4}{2\phi_1\phi_2\phi_3\phi_4}.$$

Equality of the two denominators now follows from [3, p. 488, Ex. 5] or the first line of [2, p. 21, Ex. 10] by using (1.1) and (1.2), which serve also to convert the resulting equality of the numerators to (4.8) after multiplying both sides by $\theta_p^4(0)$. \square

Example 4.1. Proving that (4.8) agrees with [3, p. 488, Ex. 4] and [2, p. 21, Ex. 10] is simple for $p = 4$ with the help of (2.21) but more complicated for $p = 2$ or 3 . In each case assume equality of the old and new expressions, convert to φ functions, and confirm equality by using (2.14) twice.

5. Addition and pseudoaddition formulas

5.1. Addition formulas

Proposition 5.1. If $\{p, q, r\} = \{2, 3, 4\}$, if twelve φ functions are defined by (1.3) and (1.4), and if Δ_{pq} is defined by (2.9) and (2.14), then

$$\varphi_{p1}(z+w) = \frac{\varphi_{p1}^2(z)\varphi_{p1}^2(w) - \Delta_{pq}\Delta_{pr}}{\varphi_{p1}(z)\varphi_{q1}(w)\varphi_{r1}(w) + \varphi_{p1}(w)\varphi_{q1}(z)\varphi_{r1}(z)}, \quad (5.1)$$

$$\varphi_{1p}(z+w) = \frac{\varphi_{1p}(z)\varphi_{qp}(w)\varphi_{rp}(w) + \varphi_{1p}(w)\varphi_{qp}(z)\varphi_{rp}(z)}{1 - \Delta_{pq}\Delta_{pr}\varphi_{1p}^2(z)\varphi_{1p}^2(w)}, \quad (5.2)$$

$$\varphi_{pq}(z+w) = \frac{\varphi_{p1}(z)\varphi_{q1}(z)\varphi_{p1}(w)\varphi_{q1}(w) + \Delta_{pq}\varphi_{r1}(z)\varphi_{r1}(w)}{\varphi_{q1}^2(z)\varphi_{q1}^2(w) + \Delta_{pq}\Delta_{qr}}. \quad (5.3)$$

Proof. With notation changed for present purposes, [1, (5.1–3)] include

$$ps(u+v) = \frac{ps^2(u)ps^2(v) - \Delta(p,q)\Delta(p,r)}{ps(u)qs(v)rs(v) + ps(v)qs(u)rs(u)}, \quad (5.4)$$

$$sp(u+v) = \frac{sp(u)qp(v)rp(v) + sp(v)qp(u)rp(u)}{1 - \Delta(p,q)\Delta(p,r)sp^2(u)sp^2(v)}, \quad (5.5)$$

$$pq(u+v) = \frac{ps(u)qs(u)ps(v)qs(v) + \Delta(p,q)rs(u)rs(v)}{qs^2(u)qs^2(v) + \Delta(p,q)\Delta(q,r)}, \quad (5.6)$$

where $\{p, q, r\} = \{c, d, n\}$ as in [1, (1.7)]. By (2.5) and (2.1) we have

$$(cs(u), ds(u), ns(u)) = (z/u)(\varphi_{21}(z), \varphi_{31}(z), \varphi_{41}(z)), \quad u = \theta_3^2(0)z. \quad (5.7)$$

The same relation holds if u, z are replaced by v, w or by $u+v, z+w$, with the ratio of the second quantity to the first being in each case $1/\theta_3^2(0)$, which we shall call h for brevity. By (2.10)–(2.12) we have also

$$(\Delta(n, d), \Delta(n, c), \Delta(d, c)) = h^2(\Delta_{43}, \Delta_{42}, \Delta_{32}). \quad (5.8)$$

By (5.7) and (5.8), (5.4) is translated as

$$h\varphi_{p1}(z+w) = \frac{h^4\varphi_{p1}^2(z)\varphi_{p1}^2(w) - h^4\Delta_{pq}\Delta_{pr}}{h^3\varphi_{p1}(z)\varphi_{q1}(w)\varphi_{r1}(w) + h^3\varphi_{p1}(w)\varphi_{q1}(z)\varphi_{r1}(z)}, \quad \{p, q, r\} = \{2, 3, 4\},$$

and cancellation of h 's leaves (5.1). Similar translation of (5.6) leads to (5.3). Note that sp and pq translate as $h^{-1}\varphi_{1p}$ and φ_{pq} because $sp = ps^{-1}$ and $pq = ps/qs$. To obtain (5.2) multiply numerator and denominator of (5.1) by $\varphi_{1p}^2(z)\varphi_{1p}^2(w)$ and then take the reciprocal. \square

It is hard to find elsewhere an equation equivalent to one of the twelve equations included in (5.1)–(5.3). As a check on Proposition 5.1 we can offer only a reduction to the duplication theorems of Proposition 4.2 when $w = z$. For example, (5.1) obviously reduces to the first equation in (4.4), and (5.2) reduces to (4.5). The denominator in (5.3) reduces to the first denominator in (4.6), while the numerator in (5.3) reduces to

$$\varphi_{p1}^2\varphi_{q1}^2 + \Delta_{pq}\varphi_{r1}^2 = \varphi_{p1}^2\varphi_{q1}^2 + \Delta_{pq}(\varphi_{p1}^2 - \Delta_{pr}) = \varphi_{p1}^2(\varphi_{q1}^2 + \Delta_{pq}) - \Delta_{pq}\Delta_{pr} = \varphi_{p1}^4 - \Delta_{pq}\Delta_{pr},$$

which is the first numerator in (4.6).

5.2. Pseudoaddition formulas

In [3, pp. 467, 487, 488] the term “addition-formulae” is used to describe reduction of the products $\theta_p(y+z)\theta_p(y-z)$, $p = 1, \dots, 4$, to functions of y and functions of z , but the authors concede that these are “not addition-theorems in the strict sense.” From the many reductions of this type listed in [2, (1.4.16–19, 23–26, 30–33)], six are shown below to exhibit symmetry in subscripts 2, 3, 4 and are simplified by using normalized theta functions.

Proposition 5.2. If ϕ_i , $i = 1, \dots, 4$, is defined by (1.1) and (1.2), and if Δ_{pq} is defined by (2.9) and (2.14), then

$$\phi_1(y+z)\phi_1(y-z) = \phi_1^2(y)\phi_p^2(z) - \phi_p^2(y)\phi_1^2(z), \quad p = 2, 3, 4, \quad (5.9)$$

$$\phi_p(y+z)\phi_p(y-z) = \phi_p^2(y)\phi_p^2(z) - \Delta_{pq}\Delta_{pr}\phi_1^2(y)\phi_1^2(z), \quad \{p, q, r\} = \{2, 3, 4\}. \quad (5.10)$$

Proof. We assume the correctness of equations taken from [2] and deal only with their conversion to normalized theta functions. By [2, (1.4.30, 23, 16)] we find

$$\theta_1(y+z)\theta_1(y-z)\theta_p^2(0) = \theta_1^2(y)\theta_p^2(z) - \theta_p^2(y)\theta_1^2(z), \quad p = 2, 3, 4.$$

Replacement of θ_1 by $\phi_1\theta_1'(0)$ and θ_p by $\phi_p\theta_p(0)$ can be followed by cancellations to leave (5.9).

By [2, (1.4.31, 25, 19)] we find

$$\theta_p(y+z)\theta_p(y-z)\theta_p^2(0) = \theta_p^2(y)\theta_p^2(z) - (-1)^p\theta_1^2(y)\theta_1^2(z), \quad p = 2, 3, 4.$$

The same replacements as before produce

$$\phi_p(y+z)\phi_p(y-z)\theta_p^4(0) = \phi_p^2(y)\phi_p^2(z)\theta_p^4(0) - (-1)^p\phi_1^2(y)\phi_1^2(z)\theta_1^4(0)^4,$$

and cancellation of $\theta_p^4(0)$ leaves

$$\phi_p(y+z)\phi_p(y-z) = \phi_p^2(y)\phi_p^2(z) - (-1)^p\theta_q^4(0)\theta_r^4(0)\phi_1^2(y)\phi_1^2(z),$$

which agrees with (5.10) by (4.7). \square

Comment: It may seem paradoxical that only one side of (5.9) depends on p , thus implying that

$$\phi_1^2(y)\phi_p^2(z) - \phi_p^2(y)\phi_1^2(z) = \phi_1^2(y)\phi_q^2(z) - \phi_q^2(y)\phi_1^2(z).$$

For reassurance divide by $\phi_1^2(y)\phi_1^2(z)$ to obtain $\varphi_{p1}^2(z) - \varphi_{p1}^2(y) = \varphi_{q1}^2(z) - \varphi_{q1}^2(y)$ and thus $\varphi_{p1}^2(z) - \varphi_{q1}^2(z) = \varphi_{p1}^2(y) - \varphi_{q1}^2(y)$, in which each side is now the constant Δ_{pq} .

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